

Comparing selfinteracting scalar fields and $R + R^3$ cosmological models

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Abstract

We generalize the well-known analogies between $m^2\phi^2$ and $R + R^2$ theories to include the selfinteraction $\lambda\phi^4$ -term for the scalar field. It turns out to be the $R + R^3$ Lagrangian which gives an appropriate model for it. Considering a spatially flat Friedman cosmological model, common and different properties of these models are discussed, e.g., by linearizing around a ground state the masses of the resp. spin 0-parts coincide. Finally, we prove a general conformal equivalence theorem between a Lagrangian $L = L(R)$, $L'L'' \neq 0$, and a minimally coupled scalar field in a general potential.

Key words: cosmology - cosmological models

1 Introduction

For the gravitational Lagrangian

$$L = (R/2 + \beta R^2)/8\pi G, \quad (1)$$

$R = R_{\text{crit}} = -1/4\beta$ is the critical value of the curvature scalar (cf. NARIAI 1973, 1974 and SCHMIDT 1986) defined by $\partial L/\partial R = 0$. In regions where $R/R_{\text{crit}} < 1$ holds, we can define $\psi = \ln(1 - R/R_{\text{crit}})$ and

$$\tilde{g}_{ij} = (1 - R/R_{\text{crit}})g_{ij}$$

and obtain a Lagrangian ($8\pi G = 1$)

$$\tilde{L} = \tilde{R}/2 - 3\tilde{g}^{ij}\psi_{;i}\psi_{;j}/4 - (1 - e^{-\psi})^2/16\beta \quad (2)$$

being equivalent to L , cf. WHITT (1984), and SCHMIDT (1986) for the version of this equivalence used here.

For $\beta < 0$, i.e., the absence of tachyons in L (1), we have massive gravitons of mass $m_0 = (-12\beta)^{-1/2}$ in L , cf. STELLE (1977). For the weak field limit, the potential in (2) can be simplified to be $\psi^2/16 \cdot \beta$, i.e., we have got a minimally coupled scalar field whose mass is also m_0 . (The superfluous factor $3/2$ in (2) can be absorbed by a redefinition of ψ .) Therefore, it is not astonishing, that all results concerning the weak field limit for both $R + R^2$ -gravity without tachyons and Einstein gravity with a minimally coupled massive scalar field exactly coincide. Of course, one cannot expect this coincidence to hold for the non-linear region, too, but it is interesting to observe which properties hold there also.

We give only one example here: we consider a cosmological model of the spatially flat Friedman type, start integrating at the quantum boundary (which is obtained by

$$R_{ijkl}R^{ijkl}$$

on the one hand, and T_{00} on the other hand, to have Planckian values) with uniformly distributed initial conditions and look whether or not an inflationary phase of the expansion appears. In both cases we get the following result: The probability p to have sufficient inflation is about $p = 1 - \sqrt{\lambda}m_0/m_{\text{Pl}}$,

i.e., $p = 99.992\%$ if we take $m_0 = 10^{-5}m_{\text{Pl}}$ from GUT and $\lambda = 64$, where e^λ is the linear multiplication factor of inflation.¹

From Quantum field theory, however, instead of the massive scalar field, a Higgs field with selfinteraction turns out to be a better candidate for describing effects of the early universe. One of the advances of the latter is its possibility to describe a spontaneous breakdown of symmetry. In the following, we try to look for a purely geometric model for this Higgs field which is analogous to the above mentioned type where $L = R + R^2$ modelled a massive scalar field.

2 The Higgs field

For the massive scalar Field ϕ we have

$$L_m = - \left(\phi_{;i} \phi^{;i} - m^2 \phi^2 \right) / 2 \quad (3a)$$

and for the Higgs field to be discussed now,

$$L_\lambda = - \left(\phi_{;i} \phi^{;i} + \mu^2 \phi^2 - \lambda \phi^4 / 12 \right) / 2 \quad (3b)$$

The ground states are defined by $\phi = \text{const.}$, $\partial L / \partial \phi = 0$, i.e., $\phi = 0$ for the scalar field, and $\phi = \phi_0 = 0$, $\phi = \phi_\pm = \pm \sqrt{6\mu^2/\lambda}$ for the Higgs field.

$$\left(+\partial^2 L / \partial \phi^2 \right)^{1/2} \quad (3)$$

is the effective mass at these points, i.e., m for the scalar field (3a), and $i\mu$ at $\phi = 0$ and $\sqrt{2}\mu$ at $\phi = \phi_\pm$ for the Higgs field (3b). To have a vanishing Lagrangian at the ground state ϕ_\pm we add a constant

$$\Lambda = -3\mu^4/2\lambda \quad (4)$$

¹Cf. BELINSKY et al. (1985) for the scalar field and SCHMIDT (1986) for $R + R^2$, resp.

to the Lagrangian (3b). The final Lagrangian reads

$$L = R/2 + L_\lambda + \Lambda \quad (5)$$

with L_λ (3b) and Λ (4).

3 The nonlinear gravitational Lagrangian

Preliminarily we direct the attention to the following fact: on the one hand, for Lagrangians (3a,b) and (5) the transformation $\phi \rightarrow -\phi$ is a pure gauge transformation, it does not change any invariant or geometric objects. On the other hand,

$$R_{ijkl} \rightarrow -R_{ijkl} \quad (6)$$

or simpler

$$R \rightarrow -R \quad (7)$$

is a gauge transformation at the linearized level only: taking

$$g_{ik} = \eta_{ik} + \epsilon h_{ik} ,$$

where

$$\eta_{ik} = \text{diag}(1, -1, -1, -1)$$

then $\epsilon \rightarrow -\epsilon$ implies (6) at the linearized level in ϵ whereas even (7) does not hold quadratically in ϵ . This corresponds to fact that the ϵ^2 -term in (2) (corresponding to the ψ^3 -term in the development of \tilde{L} in powers of ψ) is the first one to break the $\psi \rightarrow -\psi$ symmetry in (2).

Now, let us introduce the general nonlinear Lagrangian $L = L(R)$ which we at the moment only assume to be an analytical function of R . The ground states are defined by $R = \text{const.}$, i.e.,

$$L' R_{ik} - g_{ik} L/2 = 0 . \quad (8)$$

Here, $L' = \partial L / \partial R$.

3.1 Calculation of the ground states

From eq. (8) one immediately sees that $\partial L/\partial R = 0$ defines critical values of the curvature scalar. For these values $R = R_{\text{crit}}$ it holds: For $L(R_{\text{crit}}) \neq 0$ no such ground state exists, and for $L(R_{\text{crit}}) = 0$, we have only one equation $R = R_{\text{crit}}$ to be solved with 10 arbitrary functions g_{ik} . We call these ground states degenerated ones. For $L = R^2$, $R_{\text{crit}} = 0$, this has been discussed by BUCHDAHL (1962). Now, let us concentrate on the case $\partial L/\partial R \neq 0$. Then R_{ij} is proportional to g_{ij} with a constant proportionality factor, i.e., each ground state is an Einstein space

$$R_{ij} = Rg_{ij}/4, \quad (9)$$

with a prescribed constant value R . Inserting (9) into (8) we get as condition for ground states

$$RL' = 2L.$$

As an example, let L be a third order polynomial

$$L = \Lambda + R/2 + \beta R^2 + \lambda R^3/12. \quad (10)$$

We consider only Lagrangians with a positive linear term as we wish to reestablish Einstein gravity in the $\Lambda \rightarrow 0$ weak field limit, and $\beta < 0$ to exclude tachyons there.

For $\lambda = 0$ we have (independently of β !) the only ground state $R = -4\Lambda$. It is a degenerated one if and only if $\beta\Lambda = 1/16$. That implies that for usual $R + R^2$ gravity (1) ($\lambda = \Lambda = 0$) $R = 0$ is the only ground state and it is a nondegenerated one.

Now, let $\lambda \neq 0$ and $\Lambda = 0$. To get nontrivial ground states we further require $\lambda > 0$. Then, besides $R = 0$, the ground states are

$$R = R_{\pm} = \pm\sqrt{6/\lambda} \quad (11)$$

being quite analogous to those of the Higgs field (3b). The ground state $R = 0$ is not degenerated (of course, this statement is independent of λ and holds true, as one knows, for $\lambda = 0$.) To exclude tachyons, we require $\beta < 0$, then R_- is not degenerated and R_+ is degenerated if and only if $\beta = -\sqrt{6/\lambda}$.

The case $\lambda\Lambda \neq 0$ will not be considered here.

3.2 Definition of the masses

For the usual $R + R^2$ theory (1), the mass is

$$m_0 = (R_{\text{crit}}/3)^{1/2} = (-12\beta)^{1/2}.$$

But how to define the graviton's masses for the Lagrangian (10)? To give such a definition a profound meaning one should do the following: linearize the full vacuum field equation around the ground state (preferably de Sitter- or anti-de Sitter space, resp.) decompose its solutions with respect to a suitably chosen orthonormal system (a kind of higher spherical harmonics) and look for the properties of its single modes. For L (1) this procedure just gave m_0 .

A little less complicated way to look at this mass problem is to consider a spatially flat Friedman cosmological model and to calculate the frequency with which the scale factor oscillates around the ground state, from which the mass m_0 turned out to be the graviton's mass for L (1), too.

Keep in mind, 1. that all things concerning a linearization around flat vacuous space-time do not depend on the parameter λ neither for the Higgs field nor for the $L(R)$ model, and 2. that a field redefinition $R \rightarrow R^* + R_\pm$ is *not* possible like $\phi \rightarrow \phi^* + \phi_\pm$ because curvature remains absolutely present.

4 The cosmological model

Now we take as Lagrangian eq. (10) and as line element

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2). \quad (12)$$

The dot denotes d/dt and $h = \dot{a}/a$. We have

$$R = -6\dot{h} - 12h^2, \quad (13)$$

and the field equation will be obtained as follows.

4.1 The field equation

For $L = L(R)$ the variation

$$\delta \left(L\sqrt{-g} \right) / \delta g^{ij} = 0$$

gives with $L' = \partial L / \partial R$

$$L' R_{ij} - g_{ij} L / 2 + g_{ij} \square L' - L'_{;ij} = 0, \quad (14)$$

cf. e.g., NOVOTNY (1985). (Be aware of sign errors in the paper of KERNER (1982) such that the results of it are wrong. Nevertheless, his ideas are fruitful ones.) It holds

$$L'_{;ij} = L'' R_{;ij} + L'' R_{;i} R_{;j}. \quad (15)$$

With eq. (15), the trace of eq. (14) reads

$$L' R - 2L + 3L'' \square R + 3L'' R_{;k} R^{;k} = 0, \quad (16)$$

i.e., with L eq. (10)

$$-2\Lambda - R/2 + \lambda R^3/12 + 6\beta \square R + \frac{3\lambda}{2}(R \square R + R_{;k} R^{;k}) = 0.$$

Inserting eqs. (12), (13), (15) into the 00-component of eq. (14) we get the equation

$$0 = h^2/2 - \Lambda/6 - 6\beta(2h\ddot{h} - \dot{h}^2 + 6h^2\dot{h}) + 3\lambda(\dot{h} + 2h^2)(6h\ddot{h} + 19h^2\dot{h} - 2\dot{h}^2 - 2h^4). \quad (17)$$

The remaining components are a consequence of this one.

4.2 The masses

Linearizing the trace equation (16) around the flat space-time (hence, $\Lambda = 0$) gives (independently of λ , of course) $R = 12\beta\Box R$, and the oscillations around the flat space-time indeed correspond to a mass $m_0 = (-12\beta)^{-1/2}$.

Now, let us linearize the ground states (11) by inserting $\Lambda = 0$ and $R = \pm\sqrt{6/\lambda} + Z$ into eq. (16). It gives

$$Z = \left(-6\beta \mp \sqrt{27\lambda/2}\right)\Box Z,$$

and, correspondingly,

$$m_{\pm} = \left(6\beta \pm \sqrt{27\lambda/2}\right)^{-1/2}. \quad (18)$$

For $\beta \ll -\sqrt{\lambda}$, m_{\pm} is imaginary, and its absolute value differs by a factor $\sqrt{2}$ from m_0 . This is quite analogous to the $\lambda\phi^4$ -theory, cf. sect. 2. Therefore, we concentrate on discussing this range of parameters.

For the ground state for $\Lambda \neq 0$, $\lambda = 0$ we get with $R = -4\Lambda + Z$ just $Z = 12\beta\Box Z$, i.e., mass m_0 just as in the case $\lambda = \Lambda = 0$.

Let us generalize this estimate to $L = L(R)$; $R = R_0 = \text{const.}$ is a ground state if

$$L'(R_0)R_0 = 2L(R_0)$$

holds. It is degenerated if $L'(R_0) = 0$. Now, linearize around $R = R_0$: $R = R_0 + Z$. For $L''(R_0) = 0$, only $Z = 0$ solves the linearized equation, and $R = R_0$ is a singular solution. For $L''(R_0) \neq 0$ we get the mass

$$m = (R_0/3 - L'(R_0)/3L'(R_0))^{1/2} \quad (19)$$

meaning the absence of tachyons for real values m . Eq. (19) is the analogue to eq. (3).

4.3 The Friedman model

Here we only consider the spatially flat Friedman model (12). Therefore, we can discuss only de Sitter stages with $R < 0$, esp. the ground state R_+ eq. (11) does not enter our discussion but R_- does.

Now, let $\Lambda = 0$. Solutions of eq. (17) with constant values h are $h = 0$ (flat space-time) and

$$h = \frac{1}{\sqrt[4]{24\lambda}}$$

(de Sitter space-time) representing the non-degenerated ground states $R = 0$ and $R = R_- = -\sqrt{6}/\lambda$, resp. Eq. (17) can be written as

$$0 = h^2(1 - 24\lambda h^4)/2 + h\ddot{h} \left(1/m_0^2 + 18\lambda(\dot{h} + 2h^2)\right) - 6\lambda\dot{h}^3 + \dot{h}^2(45\lambda h^2 - 1/2m_0^2) + 3h^2\dot{h}(1/m_0^2 + 36\lambda h^2). \quad (20)$$

First, let us consider the singular curve defined by the vanishing of the coefficient of \ddot{h} in eq. (20) in the $h - \dot{h}$ -phase plane. It is, besides $h = 0$, the curve

$$\dot{h} = -2h^2 - 1/18\lambda m_0^2 \quad (21)$$

i.e., just the curve

$$R = 1/3\lambda m_0^2 = -4\beta/\lambda$$

which is defined by $L'' = 0$, cf. eq. (16). This value equals R_+ if $\beta = -\sqrt{3\lambda/8}$, this value we do not discuss here. Points of the curve (21) fulfil eq. (20) for

$$h = \pm 1/18\lambda m_0^3 \sqrt{3} \sqrt{1 - 1/18\lambda m_0^4}$$

only, which is not real because of $\lambda \ll m_0^4$.

Therefore, the space of solutions is composed of at least 2 connected components.

Second, for $h = 0$ we have $\dot{h} = 0$ or

$$\dot{h} = -1/12\lambda m^2. \quad (22)$$

$h = \dot{h} = 0$ implies $h\ddot{h} \geq 0$, i.e. h does not change its sign. (We know this already from MÜLLER and SCHMIDT (1985), where the same model with $\lambda = 0$ is discussed.) In a neighbourhood of (22) we can make the ansatz

$$h = -t/12\lambda m_0^2 + \sum_{n=2}^{\infty} a_n t^n$$

which has solutions with arbitrary values a_2 . That means: one can change from expansion to subsequent recontraction, but only through the “eye of a needle” (22). On the other hand, a local minimum of the scale factor never appears. Further, (22) does not belong to the connected component of flat space-time.

But we are especially interested in the latter one, and therefore, we restrict to the subset $\dot{h} > \dot{h}(\text{eq. (21)})$ and need only to discuss expanding solutions $h \geq 0$. Inserting $\dot{h} = 0$,

$$\ddot{h} = h(24\lambda h^4 - l)/(2/m_0^2 + 72\lambda h^2)$$

turns out, i.e., $\ddot{h} > 0$ for $h > 1/\sqrt[4]{24\lambda}$ only. All other points in the $h-\dot{h}$ phase plane are regular ones, and one can write $d\dot{h}/dh \equiv \ddot{h}/\dot{h} = F(h, \dot{h})$ which can be calculated by eq. (20).

For a concrete discussion let $\lambda \approx 10^2 l_{\text{Pl}}^4$ and $m_0 = 10^{-5} m_{\text{Pl}}$. Then both conditions $\beta \ll -\sqrt{\lambda}$ and $|R_-| < l_{\text{Pl}}^{-2}$ are fulfilled. Now the qualitative behaviour of the solutions can be summarized: There exist two special solutions which approximate the ground state R_- for $t \rightarrow -\infty$. All other solutions have a past singularity $h \rightarrow \infty$. Two other special solutions approximate the ground state R_- for $t \rightarrow +\infty$. Further solutions have a future singularity $h \rightarrow \infty$, and all other solutions have a power like behaviour for $t \rightarrow \infty$, $a(t)$ oscillates around the classical dust model $a(t) \sim t^{2/3}$. But if we restrict the initial conditions to lie in a small neighbourhood of the unstable ground state R_- , only one of the following three cases appears:

1. Immediately one goes with increasing values h to a singularity.
2. (As a special case) one goes back to the de Sitter stage R_- .
3. (The only interesting one) One starts with a finite l_{Pl} -valued inflationary era, goes over to a GUT-valued second inflation and ends with a power-like Friedman behaviour.

In the last case to be considered here, let $\lambda = 0$, $\Lambda > 0$ and $\beta < 0$. The analogue to eq. (20) then reads

$$0 = h^2/2 - \Lambda/6 + (2h\ddot{h} - \dot{h}^2 + 6h^2\dot{h})/2m_0^2.$$

Here, always $h \neq 0$ holds, we consider only expanding solutions $h > 0$. For $\dot{h} = 0$ we have

$$\ddot{h} = (\Lambda m_0^2/3 - m_0^2 h^2)/2h.$$

For $\ddot{h} = 0$ we have $\dot{h} > m_0^2/6$ and

$$h = (\Lambda/3 + \dot{h}^2/m_0^2)^{1/2} (1 + 6\dot{h}/m_0^2)^{-1/2}.$$

Using the methods of MÜLLER and SCHMIDT (1985) (where the case $\Lambda = 0$ has been discussed) we obtain the following result: All solutions approach the de Sitter phase $h^2 = \Lambda/3$ as $t \rightarrow \infty$. There exists one special solution

approaching $\dot{h} = -m_0^2/6$ for $h \rightarrow \infty$, and all solutions have a past singularity $h \rightarrow \infty$. For a sufficiently small value Λ we have again two different inflations in most of all models.

5 The generalized equivalence

In this section we derive a general equivalence theorem between a nonlinear Lagrangian $L(R)$ and a minimally coupled scalar field ϕ with a general potential with Einstein's theory. Instead of ϕ we take

$$\psi = \sqrt{2/3} \phi.$$

This is done to avoid square roots in the exponents. Then the Lagrangian for the scalar field reads

$$\tilde{L} = \tilde{R}/2 - 3\tilde{g}^{ij}\psi_{;i}\psi_{;j}/4 + V(\psi). \quad (23)$$

At ground states $\psi = \psi_0$, defined by $\partial V/\partial\psi = 0$ the effective mass is

$$m = \sqrt{2/3} \sqrt{\partial^2 V / \partial \psi^2}, \quad (24)$$

cf. eq. (3). The variation $0 = \delta\tilde{L}/\delta\psi$ gives

$$0 = \partial V / \partial \psi + 3 \tilde{\square} \psi / 2 \quad (25)$$

and Einstein's equation is

$$\tilde{E}_{ij} = \kappa \tilde{T}_{ij} \quad (26)$$

with

$$\kappa \tilde{T}_{ij} = 3\psi_{;i}\psi_{;j}/2 + \tilde{g}_{ij} \left(V(\psi) - \frac{3}{4} \tilde{g}^{ab} \psi_{;a} \psi_{;b} \right). \quad (27)$$

Now, let

$$\tilde{g}_{ij} = e^\psi g_{ij}. \quad (28)$$

The conformal transformation (28) shall be inserted into eqs. (25, 26, 27). One obtains from (25) with

$$\begin{aligned}\psi^{;k} &:= g^{ik}\psi_{;i} \\ \square\psi + \psi^{;k}\psi_{;k} &= -2(e^\psi\partial V/\partial\psi)/3\end{aligned}\tag{29}$$

and from (26, 27)

$$E_{ij} = \psi_{;ij} + \psi_{;i}\psi_{;j} + g_{ij} \left(e^\psi V(\psi) - \square\psi - \psi_{;a}\psi^{;a} \right). \tag{30}$$

Its trace reads

$$-R = 4e^\psi V(\psi) - 3\square\psi - 3\psi_{;a}\psi^{;a}. \tag{31}$$

Comparing with eq. (29) one obtains

$$R = R(\psi) = -2e^{-\psi}\partial \left(e^{2\psi}V(\psi) \right) / \partial\psi. \tag{32}$$

Now, let us presume $\partial R/\partial\psi \neq 0$, then eq. (32) can be inverted as

$$\psi = F(R). \tag{33}$$

In the last step, eq. (33) shall be inserted into eqs. (29, 30, 31). Because of

$$F(R)_{;ij} = \partial F/\partial R \cdot R_{;ij} + \partial^2 F/\partial R^2 \cdot R_{;i}R_{;j}$$

and $\partial F/\partial R \neq 0$, eq. (30) is a fourth order equation for the metric g_{ij} . We try to find a Lagrangian $L = L(R)$ such that the equation $\delta L\sqrt{-g}/\delta g^{ij} = 0$ becomes just eq. (30). For $L' = \partial L/\partial R \neq 0$, eq. (14) can be solved to be

$$E_{ij} = -g_{ij}R/2 + g_{ij}L/2L' - g_{ij}\square L'/L' - L'_{;ij}/L'. \tag{34}$$

We compare the coefficients of the $R_{;ij}$ terms in eqs. (30) and (34), this gives

$$\begin{aligned}\partial F/\partial R &= L''/L', \quad \text{hence} \\ L(R) &= \mu \int_{R_0}^R e^{F(x)} dx + \Lambda_0,\end{aligned}\tag{35}$$

with suitable constants Λ_0 , μ , and R_0 , $\mu \neq 0$. We fix them as follows: We are interested in a neighbourhood of $R = R_0$ and require $L'(R_0) = 1/2$. (Otherwise L should be multiplied by a constant factor.) Further, a constant translation of ψ can be used to obtain $F(R_0) = 0$, hence $\mu = 1/2$, $L(R_0) = \Lambda_0$, and

$$L'(R_0) = \partial F / \partial R(R_0) / 2 \neq 0.$$

With (35) being fulfilled, the traceless parts of eqs. (30) and (35) identically coincide. Furthermore, we have

$$\square L' / L' = \square F + F^{;i} F_{;i}$$

and it suffices to test the validity of the relation

$$e^F V(F(R)) = -R/2 + L/2L'.$$

It holds

$$\begin{aligned} 2L' &= e^F, \quad \text{i.e.,} \\ e^{2F} V(F(R)) &= L - R e^F / 2. \end{aligned} \tag{36}$$

At $R = R_0$, this relation reads $V(0) = \Lambda_0 - R_0/2$. Applying $\partial/\partial R$ to eq. (36) gives just eq. (29), and, by the way, $V'(0) = R_0/2 - 2\Lambda_0$. In sum,

$$L(R) = V(0) + R_0/2 + \int_{R_0}^R e^{F(x)} dx / 2,$$

where $F(x)$ is defined via $F(R_0) = 0$,

$$\psi = F \left(-2e^{-\psi} \partial(e^{2\psi} V(\psi)) / \partial \psi \right).$$

Now, let us go the other direction: Let $L = L(R)$ be given such that at $R = R_0$, $L'L'' \neq 0$. By a constant change of L let $L'(R_0) = 1/2$. Define $\Lambda_0 = L(R_0)$, $\psi = F(R) = \ln(2L'(R))$ and consider the inverted function $R = F^{-1}(\psi)$. Then

$$V(\psi) = (\Lambda_0 - R_0/2)e^{-2\psi} - e^{-2\psi} \int_0^\psi e^x F^{-1}(x) dx / 2 \tag{37}$$

is the potential ensuring the above mentioned conformal equivalence. This procedure is possible at all R -intervals where $L'L'' \neq 0$ holds. For analytical functions $L(R)$, this inequality can be violated for discrete values R only (or one has simply a linear function $L(R)$ being Einstein gravity with Λ -term).

Examples: 1. Let $L = \Lambda + R^2$, $R_0 = 1/4$, then $4R = e^\psi$ and

$$V(\psi) = \Lambda e^{-2\psi} - 1/16. \quad (38)$$

(For $\Lambda = 0$, this is proven in BICKNELL (1974) and STAROBINSKY and SCHMIDT (1987).)

2. Let $L = \Lambda + R/2 + \beta R^2 + \lambda R^3/12$, $R_0 = 0$, hence $\beta \neq 0$ is necessary. We get

$$\begin{aligned} e^\psi - 1 &= 4\beta R + \lambda R^2/2 \quad \text{and} \\ V(\psi) &= \Lambda e^{-2\psi} + \\ 2\beta\lambda^{-1}e^{-2\psi} &\left(e^\psi - 1 - 16\beta^2(3\lambda)^{-1}((1 + \lambda(e^\psi - 1)/8\beta^2)^{3/2} - 1) \right). \end{aligned} \quad (39)$$

The limit $\lambda \rightarrow 0$ in eq. (39) is possible and leads to

$$V(\psi) = \Lambda e^{-2\psi} - (e^{-\psi} - 1)^2/16\beta, \quad \text{cf. eq. (2).}$$

Now, let R_0 be a non-degenerated ground state, hence

$$L(R) = \Lambda_0 + (R - R_0)/2 + L''(R_0)(R - R_0)^2/2 + \dots$$

with $L''(R_0) \neq 0$ and $\Lambda_0 = R_0/4$, cf. sct. 3.1. Using eq. (37) we get $V'(0) = 0$ and

$$V''(0) = R_0/2 - 1/4L''(R_0).$$

Inserting this into eq. (24) we exactly reproduce eq. (19). This fact once again confirms the estimate (19) and, moreover, shows it to be a true analogue to eq. (3). To understand this coincidence one should note that at ground states, the conformal factor becomes a constant $= 1$.

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